GLOBAL OPTIMIZATION ALGORITHMS FOR A CLASS OF FRACTIONAL PROGRAMMING PROBLEMS

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ABSTRACT

This paper addresses the problem of minimizing a sum of fractional functions over a convex set, where each fractional function is described by the ratio between a convex and a concave function. Linear-fractional programming problems fall into this category as important special cases. Two global optimization algorithms based on a suitable reformulation of the problem in the outcome space are proposed. Global minimizers are obtained as the limit of the optimal solutions of a sequence of special indefinite quadratic programs, solved by using a constraint enumeration procedure, according with the first algorithm, and a sequence of special linear-fractional programs, solved by using a rectangular branch and bound procedure, according to the second. Both algorithms exploit the relatively small number of half-spaces needed for approximating the original problem in the outcome space. A comparison of the algorithms based on some computational experiences is reported.

KEYWORDS. Global Optimization. Fractional Programming. Convex Analysis. Main Area: Mathematical Programming.

RESUMO

Este trabalho aborda o problema de minimizar uma soma de funções fracionais sobre um conjunto convexo, sendo que cada fração é descrita como a razão entre uma função convexa e uma função côncava, ambas positivas sobre a região viável do problema. Dois algoritmos de otimização global baseados numa reformulação adequada do problema no espaço das funções são propostos. Minimizadores globais são obtidos como o limite das soluções ótimas de uma sequência de problemas quadráticos indefinidos especiais, resolvidos por meio de um procedimento de enumeração de restrições, de acordo com o primeiro algoritmo, e de uma sequência de problemas lineares-fracionais especiais, resolvidos por meio de um procedimento *branch and bound* retangular, de acordo com o segundo algoritmo. Ambos algoritmos exploram o relativamente pequeno número de semi-espaços necessários para aproximar o problema original no espaço das funções. Uma comparação entre os algoritmos baseada em algumas experiências computacionais é relatada.

PALAVRAS-CHAVE. Otimização Global. Programação Fracionária. Análise Convexa. Área Principal: Programação Matemática.

1 Introduction

This paper addresses a class of generalized fractional programming problems, in which the objective is to minimize a sum of fractional functions over some feasible region. This class of nonconvex problems mathematically describes many important applications in engineering, finance optimization, decision making, and, more specifically, in multistage stochastic shipping problems (Almogy and Levin, 1970), bond portfolio optimization (Konno and Inori, 1989), hospital management (Mathis and Mathis, 1995), cluster analysis (Rao, 1971) and certain queuing location problems (Drezner et al., 1990), among others. In particular, if each fractional function is a risk-to-profit measure, then by solving the problem one seeks a compromise solution for a (possibly weighted) sum of risk-to-profit ratios.

The generalized fractional programming problem is known as a difficult optimization problem which may have local minimizers that are not global minimizers. In this paper two approaches for globally solving generalized fractional problems are proposed and tested. Problems of the following form are considered:

$$\min_{x\in\Omega} v(x) = \min_{x\in\Omega} \sum_{i=1}^{r} \frac{p_i(x)}{q_i(x)}, \tag{1.1}$$

where $p_i : \mathbb{R}^n \to \mathbb{R},\ i = 1,2,\dots,r$, are convex and $q_i : \mathbb{R}^n \to \mathbb{R},\ i = 1,2,\dots,r$, are concave functions. It is also assumed that

$$\Omega = \{x \in \mathbb{R}^n \ : \ g_j(x) \le 0,\ j = 1,2,\dots,p\},$$

is a nonempty compact convex set and that $p_i, q_i,\ i = 1,2,\dots,r$, are positive functions over Ω. Generalized linear-fractional problems are obtained when all the functions involved in the formulation (1.1) are linear.

The literature in fractional programming has been dominated by the the analysis of problems with only one fraction since the early 1960's (Schaible, 1995). However, since the 1980's, a number of different approaches for solving the generalized fractional problem, or one of its special cases, have been proposed. The linear-fractional case has attracted special interest. In Konno et al. (1991), an algorithm for globally solving problems containing only two fractions is proposed. The general algorithm proposed in Quesada and Grossmann (1995) requires only that each numerator and each denominator is positive over the feasible region. The image (outcome) space algorithm by Falk and Palocsay (1994) is suitable for minimizing sums (or products) of fractions of linear functions.

Examples of algorithms that address the generalized fractional problem (1.1) are found in Konno and Kunno (1990) (for the case of linear fractions) and Benson (2001) (for the case of nonlinear fractions). In Konno at el. (1990), by using a parametric transformation, the authors obtain an equivalent concave minimization problem, which is then solved through a cutting plane algorithm. Benson (2001) introduces a branch and bound search procedure that globally solves the nonlinear fractional problem by concentrating primarily on solving an equivalent outcome space problem.

In this paper, two global optimization algorithms based on a suitable reformulation of the problem in the outcome space are proposed. Global minimizers are obtained as the limit of the optimal solutions of a sequence of special indefinite quadratic programs solved by using a constraint enumeration procedure, according with the first algorithm, and a sequence of special linear-fractional programs solved by using a rectangular branch and bound procedure, according to the second. Both algorithms exploit the relatively small number of half-spaces needed for approximating the original problem in the outcome space.

The paper is organized in six sections, as follows. In Section 2, the problem is reformulated in the outcome space and an outer approximation approach for solving generalized fractional problems is outlined. In Sections 3 and 4, respectively, the relaxation-constraint enumeration algorithm and the relaxation-branch and bound algorithm are derived. A comparison of the algorithms based on some computational experiences is reported in Section 5. Conclusions are presented in Section 6.

Notation. The set of all n-dimensional real vectors is represented as $\mathbb{R}^n$. The sets of all nonnegative and positive real vectors are denoted as $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. Inequalities are meant to be componentwise: given $x,y \in \mathbb{R}^n_+$, then $x \ge y$ ($x - y \in \mathbb{R}^n$) implies $x_i \ge y_i, i = 1,2,\dots,n$. Accordingly, $x > y$ ($x - y \in \mathbb{R}^n_{++}$) implies $x_i > y_i, i = 1,2,\dots,n$. The standard inner product in $\mathbb{R}^n$ is denoted as $\langle x,y\rangle$. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined on Ω, then $f(\Omega) := \{f(x) \ : \ x \in \Omega\}$. The symbol := means *equal by definition*.

2 The Outcome Space Approach

The outcome space approach for solving generalized fractional problems is inspired in a similar approach recently introduced in Oliveira and Ferreira (2010) for solving generalized multiplicative problems of the form

$$\min_{x\in\Omega} v(x) = \min_{x\in\Omega} \sum_{i=1}^{r} f_{2i-1}(x) f_{2i}(x), \tag{2.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R},\ i = 1,2,\dots,m,\ m := 2r$, are convex positive functions over a nonempty compact convex set Ω. It should be noted that although the product of any two convex positive functions is quasiconvex, the sum of quasiconvex functions is not quasiconvex, in general. Therefore, problem (2.1) may have local minimizers that are not global minimizers.

Since in the generalized fractional problem (1.1) the numerators $p_i : \mathbb{R}^n \to \mathbb{R},\ i = 1,2,\dots,r$, are convex and the denominators $q_i : \mathbb{R}^n \to \mathbb{R},\ i = 1,2,\dots,r$, are concave positive functions over Ω and, consequently, each function $1/q_i(x)$ is convex and positive over Ω, it follows that the fractional programming problem (1.1) can be reduced to the generalized multiplicative problem (2.1).

The objective function in (2.1) can be written as the composition $u(f(x))$, where $u : \mathbb{R}^m \to \mathbb{R}$ is defined by

$$u(y) := \sum_{i=1}^{r} y_{2i-1} y_{2i}.$$

The function u can be viewed as a particular aggregating function for the problem of maximizing the vector-valued objective $f := (f_1, f_2, \dots, f_m)$ over Ω (Yu, 1985). The image of Ω under f,

$$\mathcal{Y} := f(\Omega), \tag{2.2}$$

is the outcome space associated with problem (2.1). Since f is positive over Ω, it follows that u is strictly increasing over $\mathcal{Y}$ and any optimal solution of (2.1) is Pareto-optimal or efficient (Yu, 1985). It is known from the multiobjective programming literature that if $x \in \Omega$ is an efficient solution of (2.1), then there exists $w \in \mathbb{R}^m_+$ such that x is also an optimal solution of the convex programming problem

$$\min_{x\in\Omega} \langle w, f(x)\rangle. \tag{2.3}$$

Conversely, if $x(w)$ is any optimal solution of (2.3), then $x(w)$ is efficient for (2.1) if $w \in \mathbb{R}^m_{++}$. By defining

$$\mathcal{W} := \left\{ w \in \mathbb{R}^m_+ \ :\ \sum_{i=1}^{m} w_i = 1 \right\},$$

the efficient set of (2.1) can be completely generated by solving (2.3) over $\mathcal{W}$.

The outcome space formulation of problem (2.1) is simply

$$\min_{y\in\mathcal{Y}} u(y) = \min_{y\in\mathcal{Y}} \sum_{i=1}^{r} y_{2i-1} y_{2i}. \tag{2.4}$$

The solution approaches which aim at solving problem (2.1) by solving its equivalent problem (2.4) in the outcome space basically differ in the way of representing the nonconvex set $\mathcal{Y}$. In Oliveira and Ferreira (2010), a suitable representation is derived with basis on the following convex analysis result, whose proof can be found in Lasdon (1970).

Lemma 2.1. *Given $y \in \mathbb{R}^m$, the inequality $f(x) \le y$ has a solution $x \in \Omega$ if and only if y satisfies*

$$\langle w, y\rangle \ge \min_{x\in\Omega} \langle w, f(x)\rangle \quad \textit{for all} \ \ w \in \mathcal{W}, \tag{2.5}$$

or, equivalently,

$$\min_{x\in\Omega} \langle w, f(x) - y\rangle \le 0 \quad \textit{for all} \ \ w \in \mathcal{W}.$$

The above Lemma is instrumental for showing that problem (2.4) admits an equivalent formulation with a convex feasible region.

Theorem 2.2. *Let $y^\star$ be an optimal solution of problem*

$$\min_{y \in \mathcal{F}} u(y), \tag{2.6}$$

where

$$\mathcal{F} := \left\{ \underline{y} \le y \le \bar{y} : \langle w, y \rangle \ge \min_{x \in \Omega} \langle w, f(x) \rangle \ \text{for all} \ \ w \in \mathcal{W} \right\}$$

and

$$\underline{y}_i := \min_{x \in \Omega} f_i(x) > 0, \ \ \bar{y}_i := \max_{x \in \Omega} f_i(x), \ \ i = 1, 2, \ldots, m. \tag{2.7}$$

Then $y^\star$ is also an optimal solution of (2.4). Conversely, if $y^\star$ solves problem (2.4), then $y^\star$ also solves problem (2.6).

Proof. See Oliveira and Ferreira (2010). □

2.1 Relaxation Procedure

Problem (2.6) has a small number of variables, but infinitely many linear inequality constraints. An adequate approach for solving (2.6) is relaxation. The relaxation algorithm evolves by determining y^k, a global maximizer of u over an outer approximation $\mathcal{F}^k$ of $\mathcal{F}$ described by a subset of the inequality constraints (2.5), and then appending to $\mathcal{F}^k$ only the inequality constraint most violated by y^k. The most violated constraint is found by computing

$$\theta(y) \ := \max_{w \in \mathcal{W}} \phi_y(w), \tag{2.8}$$

where

$$\phi_y(w) := \min_{x \in \Omega} \langle w, f(x) - y \rangle. \tag{2.9}$$

Maximin problems as the one described by (2.8) and (2.9) arise frequently in optimization, engineering design, optimal control, microeconomic and game theory, among other areas.

Lemma 2.3. *$y \in \mathbb{R}^m$ satisfies the inequality system (2.5) if and only if $\theta(y) \le 0$.*

Proof. If $y \in \mathbb{R}^m$ satisfies the inequality system (2.5), then $\min_{x \in \Omega} \langle w, f(x) - y \rangle \le 0$ for all $w \in \mathcal{W}$, implying that $\theta(y) \le 0$. Conversely, if $y \in \mathbb{R}^m$ does not satisfy the inequality system (2.5), then $\min_{x \in \Omega} \langle w, f(x) - y \rangle > 0$ for some $w \in \mathcal{W}$, implying that $\theta(y) > 0$. □

If $\theta(y^k) > 0$, then, as a byproduct, the optimal solution of the maximin problem (2.8)-(2.9) characterizes the most violated inequality constraint. As the pointwise minimum of linear functions (indexed by $x \in \Omega$), ϕ_{y^k} is a concave function. Therefore, $\theta(y^k)$ is computed by solving a convex problem.

Some useful properties of θ and ϕ are discussed in Oliveira and Ferreira (2008, 2010). In particular, $f(x(w^0)) - y$ is a subgradient of ϕ_y at any $w^0 \in \mathcal{W}$, and the graph of ϕ_y lies on (or below) the graph of the hyperplane $\phi_y(w^0) + \langle f(x(w^0)) - y, w - w^0 \rangle$. This hyperplane is a supporting hyperplane to the hypograph of ϕ_y, which enables piecewise linear approximations for ϕ_y. A l-th approximation for ϕ_y would be

$$\phi_y^l = \min_{1 \le i \le l} \left\{ \langle w, f(x(w^i)) - y \rangle \right\}. \tag{2.10}$$

Problem (2.8) is then replaced with the problem of maximizing ϕ_y^l over $\mathcal{W}$, which in turn can be posed as the linear programming problem

$$\max_{w \in \mathcal{W}, \sigma} \ \sigma \quad \text{s. t.} \quad \sigma \le \langle w, f(x(w^i)) - y \rangle, \quad i = 1, 2, \ldots, l. \tag{2.11}$$

Let (w^{l+1}, σ^{l+1}) be the optimal solution of the linear program (2.11). If $\sigma^{l+1} - \phi(w^{l+1})$ is less than a prescribed tolerance, then $\theta(y) := \sigma^{l+1}$. Otherwise, a new subgradient $f(x(w^{l+1})) - y)$ is obtained by solving the convex problem in (2.9) and the procedure repeated.

2.2 Basic Algorithm

Consider the initial polytope

$$\mathcal{F}^0 := \left\{ y \in \mathbb{R}^m : 0 < \underline{y} \le y \le \bar{y} \right\}, \tag{2.12}$$

where $\underline{y}$ and $\bar{y}$ are defined in (2.7). The computations of $\underline{y}$ and $\bar{y}$ demand m convex and m concave minimizations. While the computation of $\underline{y}$ is relatively inexpensive, the computation of $\bar{y}$ requires the solution of m nonconvex problems. However, the usual practice of setting the components of $\bar{y}$ sufficiently large has been successfully applied.

It is readily seen that the minimization of u over $\mathcal{F}^0$ is achieved at $y^0 = \underline{y}$. The utopian point y^0 rarely satisfies the inequality system (2.5), that is, $\theta(y^0) > 0$, in general. By denoting as $w^0 \in \mathcal{W}$ the corresponding maximizer in (2.8), one concludes that y^0 is not in (most violates) the supporting negative half-space

$$\mathcal{H}^0_+ = \left\{ y \in \mathbb{R}^m \ : \ \langle w^0, y \rangle \ge \langle w^0, f(x(w^0)) \rangle \right\}. \tag{2.13}$$

An improved outer approximation for $\mathcal{F}$ is $\mathcal{F}^1 = \mathcal{H}^0_+ \cap \mathcal{F}^0$. If y^1 that minimizes u over $\mathcal{F}^1$ is also such that $\theta(y^1) > 0$, then a new supporting positive half-space $\mathcal{H}^1_+$ is determined, the feasible region of (2.6) is better approximated by $\mathcal{F}^2 = \mathcal{F}^1 \cap \mathcal{H}^1_+$, and the process repeated. At an arbitrary iteration k of the algorithm, the following relaxed program is solved:

$$\min_{y \in \mathcal{F}^k} u(y). \tag{2.14}$$

Problem (2.14) is a linearly constrained problem of the form

$$\min_{\underline{y} \le y \le \bar{y}} u(y) \quad \text{s. t.} \quad A^{(k)} y \ge b^{(k)}, \tag{2.15}$$

where $A^{(k)} \in \mathbb{R}^{k \times m}, b^{(k)} \in \mathbb{R}^k, \underline{y} \in \mathbb{R}^m$ and $\bar{y} \in \mathbb{R}^m$ characterize the matrix representation of $\mathcal{F}^k$.

3 A Relaxation-Constraint Enumeration Algorithm

A direct application of the results of the previous section leads to a relaxation-constraint enumeration algorithm for the generalized fractional problem. Defining $\tilde{q}_i := 1/q_i,\ i = 1, 2, ..., r$, problem (1.1) can be rewritten as

$$\min_{x \in \Omega} v(x) = \min_{x \in \Omega} \sum_{i=1}^{r} p_i(x) \tilde{q}_i(x), \tag{3.1}$$

which assumes the form (2.1) with the identifications $f_{2i-1} = p_i$ and $f_{2i} = \tilde{q}_i$ for $i = 1, 2, \ldots, r$. The k-th outer approximation of (1.1) in the outcome space is given by (2.15), where u is the quadratic function

$$u(y) = \frac{1}{2} y^T Q y, \tag{3.2}$$

$$Q = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad Q \in \mathbb{R}^{m \times m}.$$

As the matriz Q has r positive (equal to 1) and r negative (equal to -1) eigenvalues, it can be shown that at least r constraints will be active at any optimal solution of the indefinite quadratic problem (2.15). Due to the small number of constraints generated by the relaxation procedure, problem (2.15) can be efficiently solved by constraint enumeration (Horst et al., 1995). In addition, only combinations of at least r constraints that include the one most violated by y^k need to be considered at iteration $k+1$. The relaxation-constraint enumeration algorithm for globally solving the generalized fractional problem (1.1) assumes the structure below.

Algorithm 1

Step 0: Find $\mathcal{F}^0$ and set $k := 0$;

Step 1: Solve the indefinite quadratic problem

$$\min_{y \in \mathcal{F}^k} \frac{1}{2} y^T Q y,$$

obtaining y^k;

Step 2: Find $\theta(y^k)$ by solving the maximin subproblem (2.8)-(2.9). If $\theta(y^k) < \varepsilon$, where $\varepsilon > 0$ is a small tolerance, stop: y^k and $x(w^k)$ are ε-optimal solutions of (2.4) and (1.1), respectively. Otherwise, define

$$\mathcal{F}^{k+1} := \{y \in \mathcal{F}^k \ : \ \langle w^k, y \rangle \geq \langle w^k, f(x(w^k)) \rangle\},$$

set $k := k+1$ and return to Step 1.

The infinite and finite convergence properties of Algorithm 1 are analogous to those exhibited by the algorithm derived in Oliveira and Ferreira (2010) for generalized multiplicative programming.

4 A Rectangular Branch and Bound Algorithm

An alternative to the reduction of the generalized fractional problem (1.1) to the form (2.1) is the explicit consideration of the fractional terms. The associated outcome space formulation would be

$$\min_{(y,z) \in \mathcal{Y}} v(y,z) = \min_{(y,z) \in \mathcal{Y}} \sum_{i=1}^{r} \frac{y_i}{z_i}, \tag{4.1}$$

where

$$\mathcal{Y} := \{(y,z) \in \mathbb{R}^r \times \mathbb{R}^r \ : \ y = p(x), \quad z = q(x), \quad x \in \Omega\},$$

$p := (p_1, p_2, \ldots, p_r)$ and $q := (q_1, q_2, \ldots, q_r)$. Proceeding similarly, it can be shown that (4.1) is equivalent to the problem

$$\min_{(y,z) \in \mathcal{F}} v(y,z) = \min_{(y,z) \in \mathcal{F}} \sum_{i=1}^{r} \frac{y_i}{z_i}, \tag{4.2}$$

where

$$\mathcal{F} := \left\{(y,z) \in \mathbb{R}^r \times \mathbb{R}^r \ : \ y \geq p(x),\ z \leq q(x),\ \text{for some } x \in \Omega\right\}.$$

The corresponding version of Lemma 2.1 would be as follows.

Lemma 4.1. *$(y,z) \in \mathcal{F}$ if and only if (y,z) satisfies the semi-infinite inequality system*

$$\min_{x \in \Omega} \left\{ \sum_{i=1}^{r} w_i(p_i(x) - y_i) - \sum_{i=r+1}^{m} w_i(q_i(x) - z_i) \right\} \leq 0 \quad \textit{for all } w \in \mathcal{W}. \tag{4.3}$$

The results of Section 2 then lead to the following k-th outer approximation for problem (1.1):

$$\min_{\substack{\underline{y} \leq y \leq \bar{y} \\ \underline{z} \leq z \leq \bar{z}}} v(y,z) \quad \text{s. t.} \quad A_y^{(k)} y + A_z^{(k)} z \geq b^{(k)}, \tag{4.4}$$

where $A_y^{(k)} \in \mathbb{R}^{k \times r}$, $A_z^{(k)} \in \mathbb{R}^{k \times r}$, $b^{(k)} \in \mathbb{R}^k$, $\underline{y}, \bar{y} \in \mathbb{R}^r$ and $\underline{z}, \bar{z} \in \mathbb{R}^r$ are defined accordingly.

Let $\mathcal{R}$ denote either the initial rectangle $\mathcal{F}^0 := [\underline{y}, \bar{y}] \times [\underline{z}, \bar{z}]$, or a subrectangle of it. In each subrectangle, any feasible point of (4.4) provides an upper bound for the optimal value of (4.4). In Adjiman et al. (1995), the authors discuss a convex lower bound for the linear fractional term x/y inside a rectangular region $[x^L, x^U] \times [y^L, y^U]$, where x^L, x^U, y^L and y^U are the lower and upper bounds on x and

y, respectively. Fractional terms of the form x/y are underestimated by introducing a new variable ξ and two inequalities which depend on the bounds on x and y. By using the ideas described in Adjiman et al. (1995), a lower bound for the optimal value of (4.4) can be obtained by solving the following convex programming problem:

$$\begin{array}{ll} \min & \sum_{i=1}^{r} \xi_i \\ \text{s. t.} & A_y^{(k)} y + A_z^{(k)} z \geq b^{(k)}, \\ & \xi_i \geq \dfrac{y_i^L}{z_i} + \dfrac{y_i}{z_i^U} - \dfrac{y_i^L}{z_i^U}, \quad i = 1,2,\ldots,r, \\ & \xi_i \geq \dfrac{y_i^U}{z_i} + \dfrac{y_i}{z_i^L} - \dfrac{y_i^U}{z_i^L}, \quad i = 1,2,\ldots,r, \\ & (y,z) \in \mathcal{R}, \end{array} \tag{4.5}$$

where y_i^L, y_i^U, z_i^L and z_i^U $(i = 1,2,\ldots,r)$ are the bounds on the variables y_i and z_i in some subrectangle $\mathcal{R}$. The rectangular branch and bound algorithm for globally solving the k-th outer approximation of the generalized fractional problem (1.1) assumes the structure below. Convergence results for rectangular branch and bound algorithms can be found in Benson (2002).

Branch and Bound Algorithm

Step 0: Find $\mathcal{F}^0$ and set $k = 0$.

Step 1: Define $\mathcal{L}_0 := \{\mathcal{F}^0\}$, and let L_0 and U_0 be a lower and an upper bound for the optimal value of problem (4.4), which are found by solving problem (4.5) with $\mathcal{R} = \mathcal{F}^0$.

Step 2: While $U_k - L_k > \varepsilon$,
i) Choose $\mathcal{R} \in \mathcal{L}_k$ such that the lower bound over $\mathcal{R}$ is equal to L_k;
ii) Split $\mathcal{R}$ along one of its longest edges into $\mathcal{R}_I$ and $\mathcal{R}_{II}$;
iii) Define

$$\mathcal{L}_{k+1} := (\mathcal{L}_k - \{\mathcal{R}\}) \cup \{\mathcal{R}_I, \mathcal{R}_{II}\},$$

and L_{k+1} and U_{k+1} as the minima lower and upper bounds over all subrectangles $\mathcal{R} \in \mathcal{L}_{k+1}$.
iv) Set $k := k+1$.

5 Computational Experiments

Algorithms 1 and 2, which solve outer approximations of generalized fractional problems through constraint enumeration and branch and bound procedures, respectively, were coded in MATLAB (V. 7.0.1)/Optimization Toolbox (V. 4) and run on a personal Pentium IV system, 2.00 GHz, 2048MB RAM. The tolerances for the ε-convergences of Algorithm 1 and 2 were fixed at 10^{-5}. The tolerance for the convergence of the branch and bound algorithm was fixed at 0.05, respectively. In order to illustrate some properties of global optimization algorithms proposed, the following illustrative example discussed in Benson (2001) has been considered:

$$v(x_1,x_2) := \frac{x_1 + 3x_2 + 2}{4x_1 + x_2 + 3} + \frac{4x_1 + 3x_2 + 1}{x_1 + x_2 + 4},$$
$$\Omega := \left\{ (x_1,x_2) \mid 3x_1^2 + x_2^2 \leq 48,\ x_1 + x_2 \geq 1,\ x_1, x_2 \geq 0 \right\}.$$

It can be shown that the feasible region Ω is a nonempty compact convex set, and also that p_1, q_1, p_2 and q_2 are positive over Ω. The lower and upper bounds on $p_1, 1/q_1, p_2$ and $1/q_2$ over the feasible region are $(3, 0.0490, 4, 0.0833)$ and $(23.1660, 0.2500, 27.2298, 0.2000)$.

The global minimizer reported in Benson (2001) is $x^{Benson} = (1.000, 0.000)$, and was found by applying a bisection search which demanded 17 bisections. The corresponding optimal value of problem

(1.1) is $v^{Benson} = 1.428571$. The same global minimizer was found by Algorithm 1, in 8 iterations, and by Algorithm 2, in 5 iterations. Their convergences are reported in Tables 1 and 2, respectively. Table 3 details some aspects of the convergence of Algorithm 2, the number of branches needed and the lower and upper bounds at each outer approximation of the generalized fractional problem.

Algorithms 1 and 2 converged in 3.8 and 4.3 seconds, respectively. The CPU time of Algorithm 2 tends to increase more rapidly as the number of fractional terms, r, increase, because the computational effort demanded by the branch and bound algorithm grows exponentially with r. The influence of r in the behaviour of Algorithm 1 is less noticeable.

Table 1: Convergence of Algorithm 1

k	$y_1^k, y_2^k, y_3^k, y_4^k$	w^k	$x(w^k)$	$\theta(y^k)$
0	(3.0000,0.0489,4.0000,0.0833)	(0.3333,0.0000,0.6667,0.0000)	(0.0000,1.0000)	0.6667
1	(3.0000,0.0489,5.0000,0.0833)	(0.0043,0.0000,0.0085,0.9872)	(1.1505,0.0000)	0.1152
2	(5.0000,0.0489,4.0000,0.2000)	(0.0000,0.8708,0.1292,0.0000)	(0.1580,0.8420)	0.1724
3	(3.0000,0.0489,5.3349,0.1971)	(0.0682,0.9318,0.0000,0.0000)	(1.0840,0.0000)	0.0871
4	(4.2787,0.0489,5.3349,0.1916)	(0.0000,0.9822,0.0178,0.0000)	(1.1183,0.0000)	0.0858
5	(3.0010,0.1424,4.9995,0.2000)	(0.0000,0.9413,0.0587,0.0000)	(0.9590,0.0410)	4.7564e-004
6	(3.0000,0.1425,5.0064,0.1999)	(0.0744,0.9256,0.0000,0.0000)	(1.0272,0.0000)	3.7073e-004
7	(3.0000,0.1429,5.0000,0.2000)	(0.0383,0.0000,0.0000,0.9617)	(1.0000,0.0000)	1.3745e-006

Table 2: Convergence of Algorithm 2

k	$y_1^k, z_1^k, y_2^k, z_2^k$	w^k	$x(w^k)$	$\theta(y^k, z^k)$
0	(3.0000,20.4356,4.0000,12.0000)	(0.0000,0.5000,0.5000,0.0000)	(4.0000,0.0000)	7.2178
1	(3.0000, 6.0000,4.0000,12.0000)	(0.0833,0.0000,0.1667,0.7500)	(0.0000,1.0000)	5.4167
2	(3.0000, 7.0817,5.0889, 5.0198)	(0.8000,0.2000,0.0000,0.0000)	(1.0000,0.0000)	0.0163
3	(3.0204, 7.0817,5.0889, 5.0220)	(0.5000,0.1315,0.0000,0.5000)	(1.0000,0.0000)	8.0243e-004
4	(3.0222, 7.0817,5.0889, 5.0222)	(0.4250,0.0000,0.0300,0.5450)	(1.0000,0.0000)	-4.8601e-012

Table 3: Convergence of the Branch-and-Bound Algorithm

k	Number of branches needed	LB (lower bound)	UB (upper bound)	UB-LB
0	–	–	–	–
1	8	0.8333	0.8634	0.0301
2	7	1.4286	1.4745	0.0459
3	8	1.4286	1.4745	0.0459
4	7	1.4286	1.4745	0.0459

6 Conclusions

Two algorithms for globally solving generalized fractional problems were proposed in this paper. Additional experiences involving a number of test problems from the literature have shown that both algorithms are among the most efficient algorithms for the class of optimization problems considered. Algorithm 1 seems to be particularly attractive due to the special properties of the indefinite quadratic problem that has to be solved at each iteration. On the other hand, Algorithm 2 can be directly adapted for solving the related problem of maximizing a generalized fractional function over a convex set. This extension and specializations of the algorithms for linear-fractional problems are under current investigation by the authors.

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References

Almogy, Y. & Levin, O., *Parametric Analysis of a Multi-stage Stochastic Shipping Problem*, in J. Lawrence (ed.) Operational Research´69, Tavistock Publications, London, 359-370, 1970.
Benson, H. P. (2002), Using Concave Envelopes to Globally Solve the Nonlinear Sum of Ratios Problem, *Journal of Global Optimization*, 22, 343-364.
Benson, H. P. (2001), Global Optimization of Nonlinear Sum of Ratios, *Journal of Mathematics Analysis and Applications*, 263, 301-315.
Drezner, Z., Schaible, D. & Simchi-Levi, D. (1990), Queuing-Location Problems on the Plane, *Neval Research Logistic*, 37, 929-935.
Falk, J. E. & Palocsay, W. (1994), Image Space Analysis of Generalized Fractional Programs, *Journal of Global Optimization*, 4, 63-88.
Horst, R., Pardalos, P.M. & Thoai, N. V., *Introduction to Global Optimization*, Kluwer,Netherlands, 1995.
Konno, H. & Inori, M. (1989), Bond Portfolio Optimization by Bilinear Fractional Programming, *Journal of the Operations Research of Japan*, 22, 143-158.
Konno, H. & Konno, T. (1990), Generalized Linear Multiplicative and Fractional Programming, *Annals of Operational research*, 25, 147-161.
Konno, H., Yajima, Y., & Matsui, T. (1991), Parametric Simplex Algorithm for Solving a Special Class of Nonconvex Minimization Problems, *Journal of Global Optimization*, 1, 65-81.
Lasdon, L. S., *Optimization Theory for Large Systems*, MacMillan Publishing Co., New York, 1970.
Adjiman C.S., Dallwig, S., Floudas, C.A. & Neumaier A. (1998), A Global Optimization Method, αBB, for General Twice-Differentiable Constrained NLPs - I. Theoretical Advances, *Computers and Chemical Engineering*, 22, 1137-1158.
Mathis, F. H. & Mathis, L. J. (1995), A Nonlinear Programming Algorithm for Hospital Management, *SIAM Review*, 37, 230-234.
Oliveira, R. M. & Ferreira, P. A. V. (2008), A Convex Analysis Approach for Convex Multiplicative Programming, *Journal of Global Optimization*, 41, 579-592.
Oliveira, R. M. & Ferreira, P. A. V. (2010), An Outcome Space Approach for Generalized Convex Multiplicative Programs, *Journal of Global Optimization*, 47, 107-118.
Quesada, J. & Grossmann I. (1995), A Global Optimization Algorithm for Linear Fractional and Bilinear Programs, *Journal of Global Optimization*, 6, 39-76.
Rao, M. R. (1971), Cluster Analysis and Mathematical Programming, *Journal of the American Statistical Association*, 66, 622-626.
Schaible, S. (1995), Fractional Programming, In R. Horst and P. M. Pardalos (Editors), *Handbook of Global Optimization*, Kluwer Academic Publishers, Netherlands, 495-608.
Yu, P-L. (1985), *Multiple-Criteria Decision Making*, Plenum Press, New York.